P1 Chapter 10 :: Trigonometric Identities & Equations

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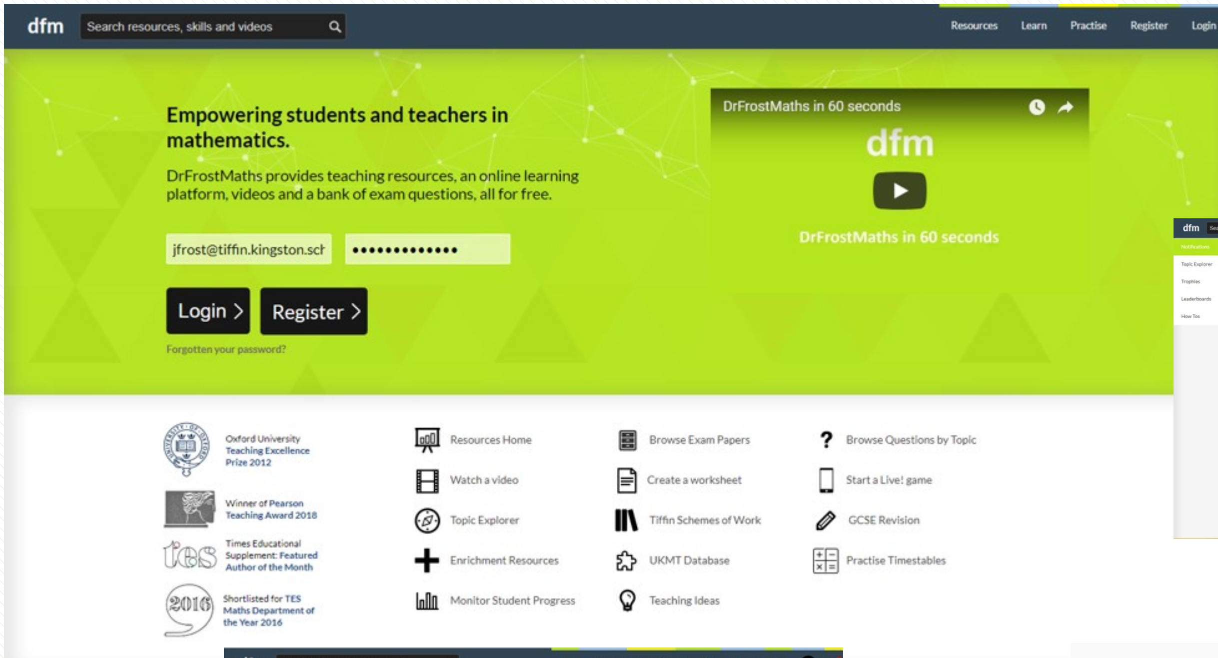

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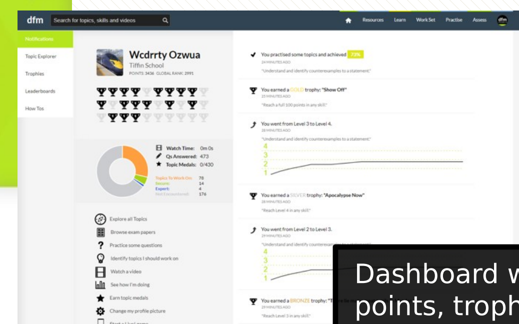

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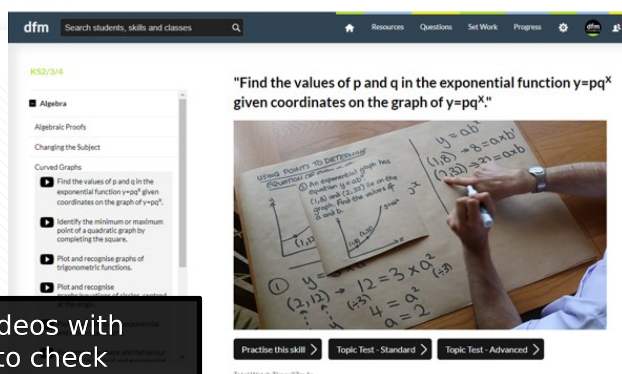
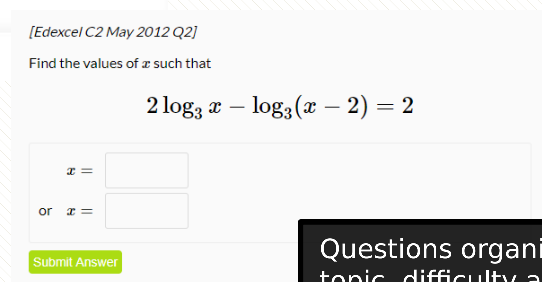




Questions organised by topic, difficulty and past paper.

Teaching videos with topic tests to check understanding.

Chapter Overview

Those who did IGCSE Further Maths or Additional Maths will be familiar with this content. Exact trigonometric values for were in the GCSE syllabus.

1:: Know exact trig values for and understand unit circle.

2:: Use identities and

Show that can be written in the form

3:: Solve equations of the form and

Solve , for .

4:: Solve equations which are quadratic in sin/cos/tan.

Solve, for , the equation

sin/cos/tan of

You will frequently encounter angles of in geometric problems.

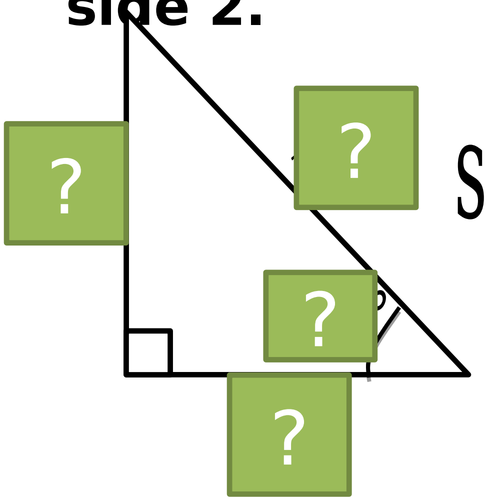
?

We see these angles in equilateral triangles and half squares.

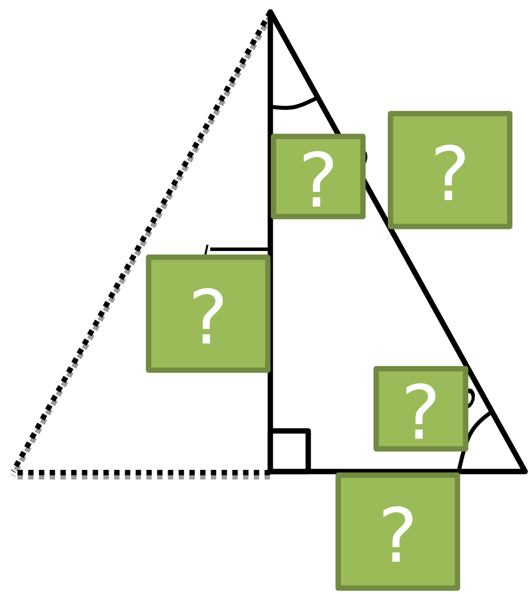
Although you will always have a calculator, you need to know how to derive these.

All you need to remember:

! Draw half a unit square and half an equilateral triangle of side 2.



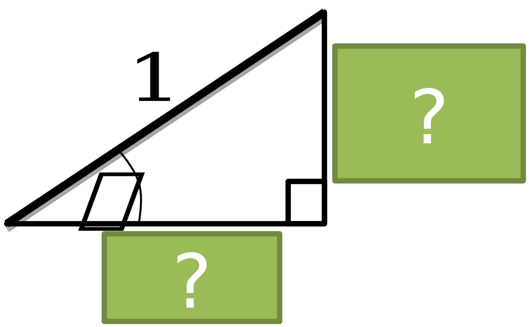
$\sin(45°) =$? ? ?



$\sin(30°)$? ? ? ? ? ?

The Unit Circle and Trigonometry

For values of in the range , you know that and are lengths on a right-angled triangle:

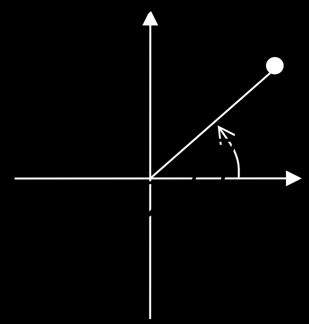


And what would be the **gradient** of the bold

?

But how do we get the rest of the graph for and when ?

The point on a unit circle, such that makes an angle with the positive -axis, has coordinates . has gradient .



Angles are always measured **anticlockwise**.
(Further Mathematicians will encounter the same when they get to Complex Numbers)

We can consider the coordinate as increases from 0 to …

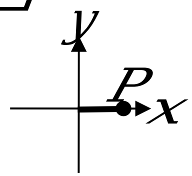
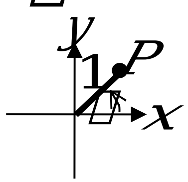
Mini-Exercise

Use the unit circle to determine each value in the table, **using either “0”, “+ve”, “-ve”, “1”, “-1” or “undefined”**. Recall that the point on the unit circle has coordinate and has gradient .

-value

-value

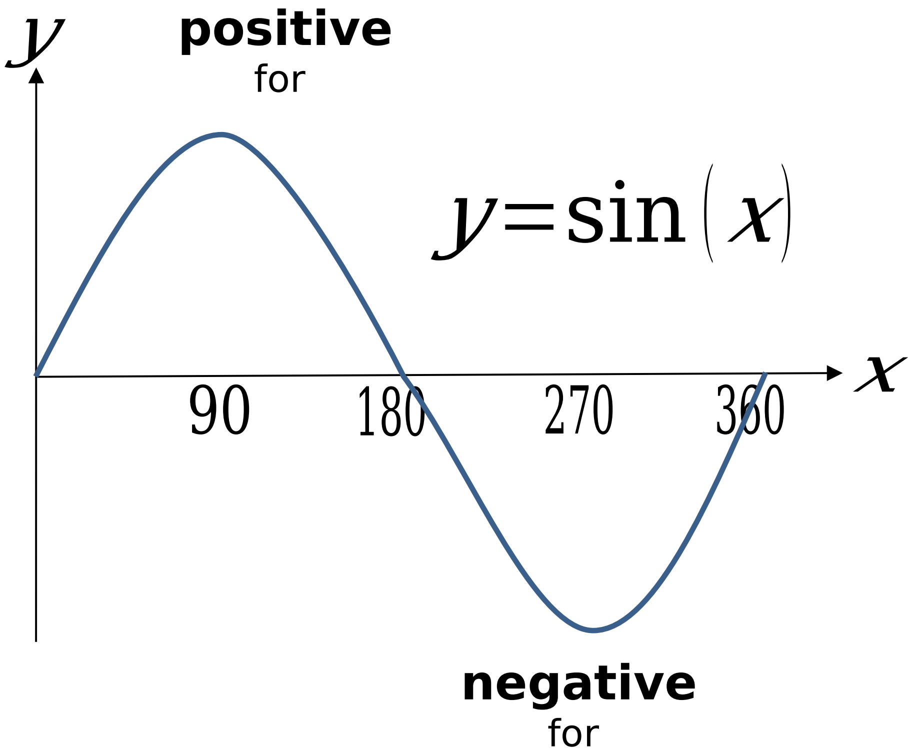
Gradient of .

	$\cos\theta$	$\sin\theta$	$\tan\theta$
$\theta=0$ 	1	0	0
$0<\theta<90°$ 	?		
$\theta=90°$?	?		
$90°<\theta<180°$?	?		

	$\cos\theta$	$\sin\theta$	$\tan\theta$
$\theta=180°$?	?		
$180°<\theta<270°$?	?		
$\theta=270°$?	?		
$270°<\theta<360°$?	?		

The Unit Circle and Trigonometry

The unit circles explains the behaviour of these trigonometric graphs beyond .
However, the easiest way to remember whether are positive or negative is to just do a **very quick sketch (preferably mentally!)** of the corresponding graph.



Note: The textbook uses something called ‘*CAST diagrams*’. I will not be using them in these slides, but you may wish to look at these technique as an alternative approach to various problems in the chapter.

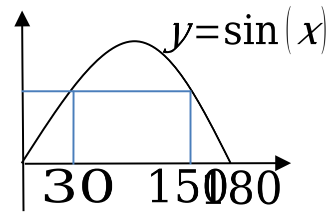
A Few Trigonometric Angle Laws

The following are all easily derivable using a quick sketch of a trigonometric graph, and are merely a <u>convenience</u> so you don't always have to draw out a graph every time.

You are highly encouraged to **memorise these** so that you can do exam questions faster.

1 $\sin(x) = \sin(180° - x)$

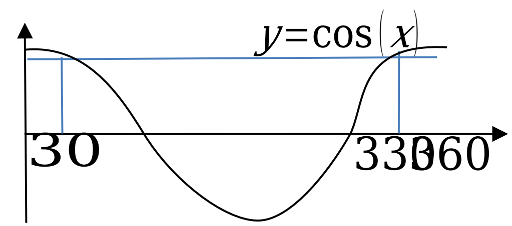
e.g.



We saw this in the previous chapter when covering the 'ambiguous case' when using the sine rule.

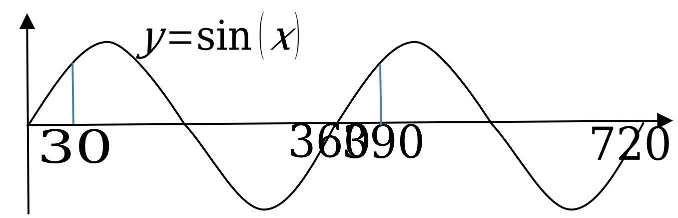
2 $\cos(x) = \cos(360° - x)$

e.g.



3 and repeat every
but every

e.g.



4

e.g.

Remember from the previous chapter that "cosine" by definition is the sine of the "complementary" angle.
This was/is never covered in the textbook but caught everyone by surprise when it came up in a C3 exam.

Examples

Without a calculator, work out the value of each below.

?

?

?

?

?

?

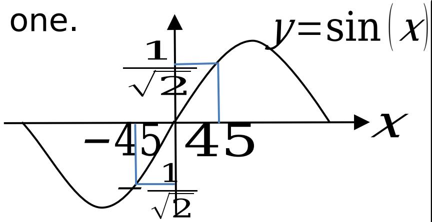
?

repeats every so can subtract

For we can subtract from .

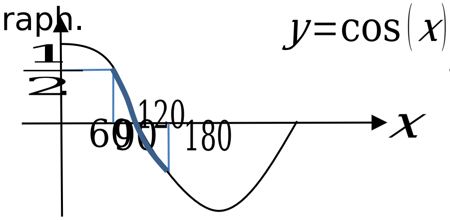
For we can subtract from .

We have to resort to a sketch for this one.



cos repeats every .

Again, let's just use a graph.



The graph is rotationally symmetric about . Since is above , we get the same value for , except negative.

Use the 'laws' where you can, but otherwise just draw out a quick sketch of the graph.

- repeat every but every

Froflections: It's not hard to see from the graph that in general, . Even more generally, a function is known as an '**odd function**' if . is similarly 'odd' as .

A function is **even** if . Examples are and as and . You do not need to know this for the exam.

Test Your Understanding

Without a calculator, work out the value of each below.

?

?

?

?

?

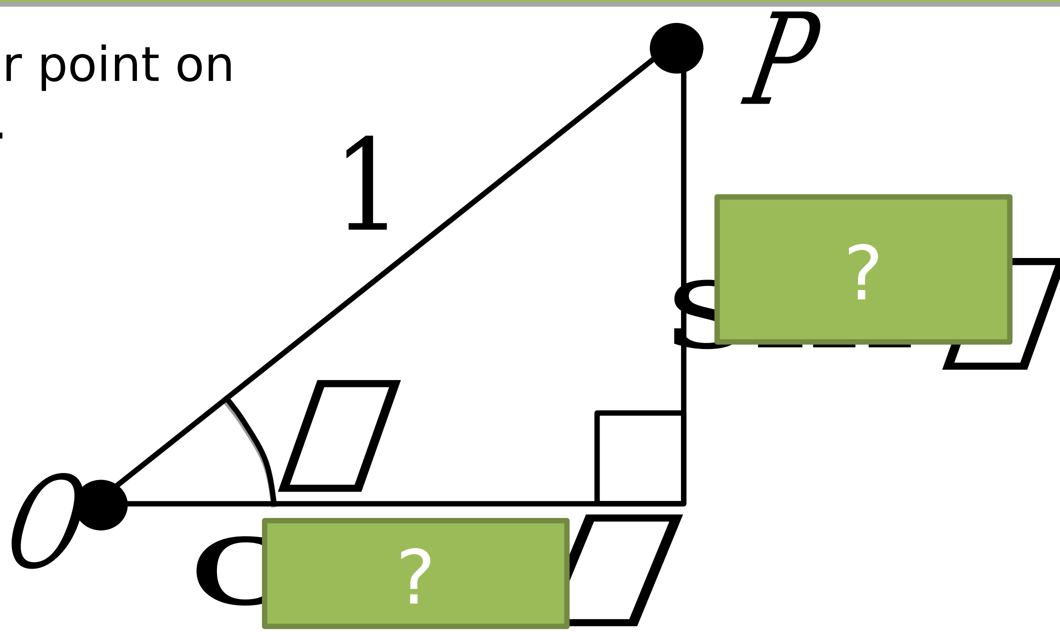
- repeat every but every

Exercise 10A/B

Pearson Pure Mathematics Year 1/AS
Page 207, 209

Trigonometric Identities

Returning to our point on the unit circle...



1 **Then**

?

2

Pythagoras gives you...

?

You are really uncool if you get

we can square!

Application of identities #1: Proofs

Prove that

?

?

?

?

Recall that means 'equivalent to', and just means the LHS is <u>**always**</u> equal to the RHS for all values of .
From Chapter 7 ('Proofs') we saw that usually the best method is to manipulate one side (e.g. LHS) until we get to the other (RHS).

More Examples

Edexcel C2 June 2012 Paper 1 Q16

Prove that

?

?

?

?

Fro Tip #2: In any addition/subtraction involving at least one fraction (with trig functions), always combine algebraically into one.

Simplify

?

Fro Tip #3: Look out for and . Students often don't spot that these can be simplified.

Test Your Understanding

Prove that

?

Prove that

?

AQA IGCSE Further Maths Worksheet

Prove that

?

Exercise 10C

Pearson Pure Mathematics Year 1/AS
Page 211-212

Extension:

[MAT 2008 1C] The simultaneous equations in ,

are solvable:

A) for all values of in range

B) except for one value of in range

C) except for two values of in range

D) except for three values of in range

?

Solving Trigonometric Equations

Remember those trigonometric angle laws (on the right) earlier this chapter? They're about to become **super freakin' useful**!

Reminder of 'trig laws':

- repeat every but every

Solve in the interval .

?

Froculator Note:
When you do , and on a calculator, it gives you only one value, known as the **principal value**.

Solve in the interval

?

Fro Tip: Look out for the solution range required. is a particularly common one.

repeats every , so can add/subtract as we please.

Slightly Harder Ones...

Solve in the interval .

?

Solve in the interval .

?

Hint: The problem here is that we have two different trig functions. Is there anything we can divide both sides by so we only have one trig function?

Test Your Understanding

Solve in the interval .

?

Solve in the interval .

?

Exercise 10D

Pearson Pure Mathematics Year 1/AS
Page 215-216

Solve in the interval .

?

Solve in the interval .

?

Solve in the interval .

?

Harder Equations

Harder questions replace the angle with a linear expression.

Solve in the interval .

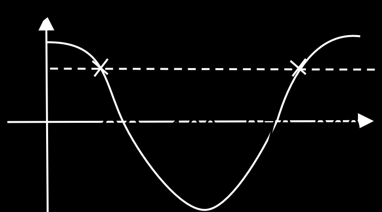
?

STEP 1: Adjust the range of values for to match the expression inside the cos.

STEP 2: Immediately after applying an inverse trig function (and BEFORE dividing by 3!), find all solutions up to the end of the interval.

STEP 3: Then do final manipulation to each value.

As mentioned before, in general you tend to get a pair of values per (for any of sin/cos/tan), except for or :



Thus once getting your first pair of values (e.g. using or to get the second value), keep adding to generate new pairs.

Further Examples

Solve in the interval .

?

Solve in the interval

?

Test Your Understanding

Edexcel C2 Jan 2013 Q4

Solve, for $0 \leq x < 180°$,

$$\cos(3x - 10°) = -0.4,$$

giving your answers to 1 decimal place. You should show each step in your working.

(7)

?

Solve in the interval .

?

Solve in the interval .

?

Solve in the interval .

?

Exercise 10E

2 Solve the following equations in the interval given:

a $\tan(45° - \theta) = -1$, $0 \leqslant \theta \leqslant 360°$

b $2\sin(\theta - 20°) = 1$, $0 \leqslant \theta \leqslant 360°$

c $\tan(\theta + 75°) = \sqrt{3}$, $0 \leqslant \theta \leqslant 360°$

d $\sin(\theta - 10°) = -\frac{\sqrt{3}}{2}$, $0 \leqslant \theta \leqslant 360°$

e $\cos(50° + 2\theta) = -1$, $0 \leqslant \theta \leqslant 360°$

f $\tan(3\theta + 25°) = -0.51$, $-90° < x \leqslant 180°$

3 Solve the following equations in the interval given:

a $3\sin 3\theta = 2\cos 3\theta$, $0 \leqslant \theta \leqslant 180°$

b $4\sin(\theta + 45°) = 5\cos(\theta + 45°)$, $0 \leqslant \theta \leqslant 450°$

c $2\sin 2x - 7\cos 2x = 0$, $0 \leqslant x \leqslant 180°$

d $\sqrt{3}\sin(x - 60°) + \cos(x - 60°) = 0$, $-180° \leqslant x \leqslant 180°$

Quadratics in sin/cos/tan

We saw that an equation can be ‘quadratic in’ something, e.g. is ‘quadratic in ’, meaning that could be replaced with another variable, say , to produce a quadratic equation .

Solve in the interval .

Method 1: Use a substitution

?

Method 2: Factorise without substitution.

?

More Examples

Solve in the interval .

?

Solve in the interval .

?

Tip: We have an identity involving and , so it makes sense to change the squared one that would match all the others.

1 Solve for θ, in the interval $0 \leqslant \theta \leqslant 360°$, the following equations.
Give your answers to 3 significant figures where they are not exact.

a $4\cos^2\theta = 1$ **b** $2\sin^2\theta - 1 = 0$ **c** $3\sin^2\theta + \sin\theta = 0$

d $\tan^2\theta - 2\tan\theta - 10 = 0$ **e** $2\cos^2\theta - 5\cos\theta + 2 = 0$ **f** $\sin^2\theta - 2\sin\theta - 1 = 0$

g $\tan^2 2\theta = 3$

Hint In part **e**, only one factor leads to valid solutions.

2 Solve for θ, in the interval $-180° \leqslant \theta \leqslant 180°$, the following equations.
Give your answers to 3 significant figures where they are not exact.

a $\sin^2 2\theta = 1$ **b** $\tan^2\theta = 2\tan\theta$

c $\cos\theta(\cos\theta - 2) = 1$ **d** $4\sin\theta = \tan\theta$

Test Your Understanding

Edexcel C2 Jan 2010 Q2

(a) Show that the equation

$$5 \sin x = 1 + 2 \cos^2 x$$

can be written in the form

$$2 \sin^2 x + 5 \sin x - 3 = 0.$$

(2)

(b) Solve, for $0 \leq x < 360°$,

$$2 \sin^2 x + 5 \sin x - 3 = 0.$$

(4)

?

Exercise 10F

Pearson Pure Mathematics Year 1/AS
Page 221-222

Extension

1 *[MAT 2010 1C]* In the range , the equation

Has how many solutions?

?

2 *[MAT 2014 1E]* As varies over the real numbers, the largest value taken by the function
 equals what?

?